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OPTIMAL STRENGTH OF A COMPOUND COLUMN

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Abstract—A column of fixed length and variable cross section consists of two homogeneous and isotropic components. The components are joined along their side surfaces and have different Young's moduli, but the same Poisson's ratio. One of the components encloses the other that has the smaller Young's modulus. For different values of the ratio of the moduli, the shape of the column, which has the largest critical buckling load under axial thrust, is determined, assuming that the volumes of the components are prescribed. The problem is solved for the case of pinned ends.

It appears that the solution of the most general problem, in which each of the areas of the component cross sections may be varied, is a combination of the solutions of some more elementary problems. Therefore, two types of problems are discussed: the compound bar with an inner component of fixed cross section and the general compound bar.

The method of solution may be extended to other boundary conditions.

1. INTRODUCTION

In this paper we consider the shape-optimization of a column that has the largest critical buckling load under axial thrust. The column consists of two prismatic, homogeneous and isotropic components, one enclosing the other and the length of the column and the volumes of each of the components are specified. Shape-optimization is concerned with the determination of the form of the cross section and its variation along the length. It is well known that the strongest homogeneous column has an equilateral triangle as cross section, if compared with any other corresponding column. Hence we shall limit our considerations to the distribution of the cross sections of the components along the length. We assume the cross sections to be convex, for each of the components similar, and the forms prescribed.

The first who solved the corresponding problem for the simple column was Clausen[1], already in 1851. Clausen's result, nearly forgotten, has been obtained independently and extended considerably by Keller[2] in 1960. Keller's work opened a new period of interest in this kind of optimization problems. For our work another three papers are of interest: an article by Tadjbakhsh and Keller[3], in which other boundary conditions are discussed than in[2], a paper by Frauenthal[4], who considered the problem with a constraint placed upon the maximum allowable prebuckling stress and a short note by Taylor[5], who treats the problem by way of an energy approach. In[4] it is also shown that the constraint on the maximum allowable stress is equivalent to a constraint on the minimum allowable gage. All these papers deal with the simple column.

The discussion of the compound column leads to the consideration of one more elementary problem, because, in the most general case, the optimal shape consists of parts in which two, one or none of the areas of the components vary along the length. Hence we shall discuss two problems separately: the compound bar with an inner component of fixed cross section and the general compound bar, in which both areas may vary. Especially the first problem will be given due attention.

The problem for the compound column presents some difficulties. If we assume that the components are joined along their side surfaces, a simple, one-dimensional stress distribution can only be obtained for the case of equal Poisson's ratios. To simplify the calculations we limit our considerations to bars, consisting of materials of which only the Young's moduli are different. The mathematics of the problem will further be simplified by considering only the case of coinciding centres of gravity of the cross sections of the components. This is owing to the fact that for the more general case the local bending stiffness is not only a function of Young's moduli and the areas of the cross sections of the components, but also of the location of the "reduced centre of gravity" (cf.[6]). For the problems under discussion, the dependence of the location drops out. There is still another limitation to be laid on the class of cross sections. We shall assume that the principal axes of the cross sections of the components have, in the straight position, constant directions with respect to a fixed coordinate system and are parallel.

In spite of the restrictions we have to put on the form of the cross sections, there are still many types that satisfy all the conditions. Among these, there are cross sections which are similar to the partial cross sections of the inner bar, but also dissimilar ones with sufficient symmetry. In Fig. 1 we show some examples.

In this paper we only discuss one boundary value problem, that of the bar with simply supported ends. We shall indicate how the method may be generalized to the consideration of other boundary value problems.

The method that will be presented in this paper, is fundamentally different from the methods, as have been proposed in this field before. While, either the differential equation is attacked directly, or a variational principle is formulated that is, or is equivalent to, an energy principle, we formulate an optimization problem for the volume at given buckling load, under the constraints for minimum and maximum allowable area. The differential equation itself is also treated as a constraint. The advantage of our method, which is closely related to methods, used in control theory, is that the mathematical operations are more simple, the disadvantage is, that it only yields the correct results if there is a monotonous relationship between volume and buckling load. Although this may be expected to be the case, in a general problem it has to be confirmed by inspection of the results, or it must be proved by mathematical means, as has been done in Section 5.

It is believed that the introduction of a method, related to those of control theory, into the field of structural optimization, is important, because this method is very powerful and can be applied to more complicated problems.

Applied to the problem under discussion, the results are exactly as can be expected: if we increase the weaker component of the column, the optimal buckling load decreases. However, as simple as this may be predicted qualitatively, the quantitive relations can only be obtained by reducing the problem to a set of transcendental equations, which must be solved on the computer.

In this paper we treat the special problem, in which the inner component is the weaker one. It seems, that the corresponding problem with the stronger component inside, is of some interest in biomechanics.

2. PRELIMINARY REMARKS

We consider the buckling of a column, the cross sections of which vary along the length, but remain similar. The column consists of two components of materials with Young's moduli E_1 and E_2 and equal Poisson's ratio. The cross section R of the bar consists of two regions R_1 and R_2, where R_2 encloses R_1. The coordinates of the points in R are measured in a Cartesian coordinate system $0xy$. We define the coordinates $\underline{r}_0$ of the "reduced centre of gravity" (see [6]) by

$$E_0 A \underline{r}_0 = \int_R E(x, y)\underline{r}\, dx\, dy = E_1 \int_{R_1} \underline{r}\, dx\, dy + E_2 \int_{R_2} \underline{r}\, dx\, dy, \tag{2.1}$$

where A is the area of the complete cross section, $E(x, y)$ denotes Young's modulus at a given point $\underline{r}$, and E_0 is given by

$$AE_0 = \int_R E(x, y)\, dx\, dy = E_1 A_1 + E_2 A_2, \tag{2.2}$$

with A_1 and A_2 the areas of the cross sections of the components, occupying R_1 and R_2, respectively.

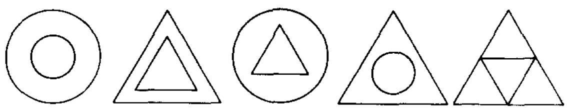

Fig. 1. Admissible cross sections.

For the problem under discussion we have

$$\underline{r}_0 = \frac{E_1 A_1 \underline{r} + E_2 A_2 \underline{r}_2}{E_1 A_1 + E_2 A_2}, \tag{2.3}$$

where $\underline{r}_1$ and $\underline{r}_2$ denote the coordinates of the centroids of the regions R_1 and R_2.

We place the origin of the coordinate system $0xy$ in the reduced centre of gravity and only discuss the case

$$\underline{r}_0 = \underline{r}_1 = \underline{r}_2 = 0. \tag{2.4}$$

Further the axes $0x$ and $0y$ coincide with the principal axes of inertia for both regions, hence we have

$$\int_{R_1} E_1 xy \,\mathrm{d}x\,\mathrm{d}y = \int_{R_2} E_2 xy \,\mathrm{d}x\,\mathrm{d}y = 0. \tag{2.5}$$

The reduced moment of inertia S about the axis $0y$ is

$$S = \int_R E(x,y) x^2 \,\mathrm{d}x\,\mathrm{d}y = E_1 \int_{R_1} x^2 \,\mathrm{d}x\,\mathrm{d}y + E_2 \int_{R_2} x^2 \,\mathrm{d}x\,\mathrm{d}y = E_1 J_1 + E_2 J_2, \tag{2.6}$$

where J_k has been defined by

$$J_k = \int_{R_k} x^2 \,\mathrm{d}x\,\mathrm{d}y, \quad k = 1, 2. \tag{2.7}$$

As we take the origin in R_1, we have

$$J_1 = k_1 A_1^2, \tag{2.8}$$

where k_1 is a constant, dependent upon the shape of R_1, but independent of A_1. For J_2 we find

$$J_2 = \int_{R_2} x^2 \,\mathrm{d}x\,\mathrm{d}y = k_2 A^2 - k_1 A_1^2, \tag{2.9}$$

where also k_2 is a constant. With (2.8) and (2.9), (2.6) becomes

$$S = k_2 E_2 (A^2 - s A_1^2), \tag{2.10}$$

with

$$s = \frac{k_1}{k_2}\left(1 - \frac{E_1}{E_2}\right). \tag{2.11}$$

We take $E_1 < E_2$.

If R and R_1 are similar, we have

$$k_1 = k_2, \quad s < 1. \tag{2.12}$$

If R and R_1 are dissimilar, we may have

$$k_1 > k_2 \tag{2.13}$$

and eventually

$$s > 1, \tag{2.14}$$

be a minimum value, to be prescribed for A ensures that the inequalities

$$J_2 > 0, \quad S > 0 \tag{2.15}$$

continue to hold. For the problem of the compound bar with proportionally varying areas, we have

$$A = kA_1, k > 1, \tag{2.16}$$

with the constant k independent of A_1. It follows from (2.10) that for this case the optimization problem reduces to that of the homogeneous bar.

3. THE STRESS DISTRIBUTION

We take the z-axis of the coordinate system along the line of centroids of the bar, which is loaded by the compressive forces P, applied to the centroids of the cross sections at the ends. As a consequence of the restrictions, put on the form of the admissible cross sections, the only stress component that is unequal to zero is t_{zz}. If there is no bending, we have a piecewise-constant stress distribution

$$t_1 = -\frac{E_1 P}{AE_0}, \quad \text{in } R_1, \tag{3.1}$$

$$t_2 = -\frac{E_2 P}{AE_0}, \quad \text{in } R_2, \tag{3.2}$$

where the value of P is taken positive and t_1 and t_2 are the values of t_{zz} in the corresponding regions. The contribution of (3.1) and (3.2) to the moment is equal to zero:

$$\int_R x t_{zz} \, dx \, dy = 0. \tag{3.3}$$

Hence we will neglect (3.1) and (3.2), as well as the very small decrease in length of the bar. As soon as buckling occurs there is another stress distribution, superimposed on (3.1) and (3.2) and given by

$$t_1 = -\frac{E_1 M}{S} x, \quad \text{in } R_1, \tag{3.4}$$

$$t_2 = -\frac{E_2 M}{S} x, \quad \text{in } R_2, \tag{3.5}$$

where M is the external moment, satisfying

$$M = -\int_R t_{zz} x \, dx \, dy. \tag{3.6}$$

If u is the displacement of the line of centroids, the equation for the buckling of the column becomes

$$\frac{d^2}{dz^2}\left(S \frac{d^2 u}{dz^2}\right) + P \frac{d^2 u}{dz^2} = 0. \tag{3.7}$$

If the column is pinned at its ends, the boundary conditions are

$$u = \frac{d^2 u}{dz^2} = 0, \quad z = 0, l \tag{3.8}$$

where l denotes the length of the bar. With (3.8), (3.7) may be integrated and reduced to

$$S\frac{d^2u}{dz^2}+Pu=0, \tag{3.9}$$

an equation that we write as

$$u''+\frac{P}{S}u=0, \tag{3.10}$$

where the primes denote differentiation with respect to z.

4. THE OPTIMIZATION PROBLEMS

(a) *Inner cross section constant*

First we consider the compound bar, with an inner component of fixed cross section. With the aid of (2.10), we write (3.10) in the form

$$u''+\frac{P/E_2k_2}{A^2-sA_1^2}u=0, \tag{4.1}$$

where A_1 is constant. The boundary conditions for this problem are

$$u=0, \quad z=0, l. \tag{4.2}$$

We prescribe the volume V of the column and the bounds p_1 and p_2 of the area A:

$$\int_0^l A\,dz=V, \tag{4.3}$$

$$p_1\leqslant A\leqslant p_2. \tag{4.4}$$

From (4.4) and the conditions of the problem we have

$$A_1l\leqslant p_1l\leqslant V\leqslant p_2l. \tag{4.5}$$

Introducing α by

$$\alpha^2=sA_1^2>0, \tag{4.6}$$

(4.1) becomes

$$u''+\frac{P/E_2k_2}{A^2-\alpha^2}u=0. \tag{4.7}$$

We shall show in Section 5 that the optimal P can be found from an optimization problem for A with subsidiary conditions. We consider the stationary value of the functional

$$\int_0^l\left[A+\lambda_1\left(u''+\frac{P/E_2k_2}{A^2-\alpha^2}u\right)+\lambda_2(q^2+A^2-(p_1-p_2)A+p_1p_2)\right]dz, \tag{4.8}$$

where λ_1 and λ_2 are Lagrangian multiplier functions and q is an auxiliary variable. We first introduce non-dimensional quantities by

$$A=\bar{A}l^2, u=\bar{u}l, z=\bar{z}l, P=\bar{P}E_2k_2l^2, q=\bar{q}l^2, p_{1,2}=\bar{p}_{1,2}l^2,$$
$$V=\bar{V}l^3, \alpha=\bar{\alpha}l^2, \bar{\lambda}_1l^3=\lambda_1, \bar{\lambda}_2=\lambda_2l^2$$

and the variational principle becomes (omitting the bars)

$$\delta\int_0^1\left[A+\lambda_1\left(u''+\frac{P}{A^2-\alpha^2}u\right)+\lambda_2(q^2+A^2-(p_1-p_2)A+p_1p_2)\right]dz=0. \tag{4.10}$$

We now consider the special (but important) case

$$s < 1, \tag{4.11}$$

for which we take

$$A \geqslant A_1 > \alpha, \tag{4.12}$$

i.e. we drop the maximum constraint p_2 for A and take

$$p_1 = A_1. \tag{4.13}$$

The variational principle now simplifies to

$$\delta \int_0^1 \left[A + \lambda_1 \left(u'' + \frac{P}{A^2 - \alpha^2} u \right) + \lambda_2 (q^2 - A + A_1) \right] \mathrm{d}z = 0. \tag{4.14}$$

Taking small variations of the field quantities A, u and q, integrating one term by parts and using the boundary conditions (4.2), we obtain

$$\int_0^1 \left[\delta A \left(1 - \lambda_1 \frac{Pu}{(A^2 - \alpha^2)^2} 2A - \lambda_2 \right) + \delta u \left(\lambda_1'' + \frac{P}{A^2 - \alpha^2} \lambda_1 \right) + 2q\lambda_2 \delta q \right] \mathrm{d}z = 0,$$

if we put

$$\lambda_1 = 0 \quad \text{for} \quad z = 0, 1. \tag{4.16}$$

From (4.15) we derive

$$\lambda_1'' + \frac{P}{A^2 - \alpha^2} \lambda_1 = 0, \tag{4.17}$$

$$\lambda_2 q = 0, \tag{4.18}$$

$$\lambda_1 \frac{2PuA}{(A^2 - \alpha^2)^2} + \lambda_2 - 1 = 0. \tag{4.19}$$

Comparing (4.7), (4.2) with (4.17) and (4.16) respectively, we see that λ_1 is proportional to u. Because the boundary value problems for u and λ_1 are homogeneous, we always may take

$$\lambda_1 = u \tag{4.20}$$

With (4.20), (4.19) becomes

$$\frac{2Pu^2 A}{(A^2 - \alpha^2)^2} + \lambda_2 - 1 = 0. \tag{4.21}$$

From (4.18) we conclude that either

$$\lambda_2 = 0, \quad q \neq 0 \tag{4.22}$$

or

$$\lambda_2 \neq 0, \quad q = 0. \tag{4.23}$$

If (4.22) holds, A is variable and the relation between u and A follows from (4.21) by putting $\lambda_2 = 0$. If (4.23) holds, A takes its minimum value, in this problem A_1.

As a consequence of the boundary conditions at $z=0$ and $z=1$, we have there

$$\lambda_2 = 1, \quad \text{for } z = 0, 1 \tag{4.24}$$

and

$$A = A_1. \tag{4.25}$$

The function λ_2 is continuous, hence (4.25) will hold over intervals near $z=0$ and $z=1$. Assuming a symmetric form for the optimal shape we have

$$A = A_1, \quad 0 \leqslant z \leqslant d; 1-d \leqslant z \leqslant 1, \tag{4.26}$$

where d is an unknown of the theory. At $z=d$ we have $\lambda_2 = 0$ and thus

$$u = (2P)^{-1/2}A^{-1/2}(A^2-\alpha^2), \tag{4.27}$$

which equality, we assume to hold over the whole interval $d \leqslant z \leqslant 1-d$. In this interval we find the following differential equation for A, by eliminating u from (4.7) and (4.27)

$$A''\left(A+\frac{\alpha^2}{3}A^{-1}\right)+\tfrac{1}{2}(A')^2(1-\alpha^2A^{-2})+\frac{2P}{3}=0, \quad (d \leqslant z \leqslant 1-d). \tag{4.28}$$

Using

$$A' = 0, \quad \text{for } z = \tfrac{1}{2}, \tag{4.29}$$

we derive from (4.28)

$$A' = \left(\frac{4P}{3}\right)^{1/2}\left(A^2+\frac{\alpha^2}{3}\right)^{-1}A^{3/2}\left\{\left(A_0-\frac{\alpha^2}{3}\frac{1}{A_0}\right)-\left(A-\frac{\alpha^2}{3}\frac{1}{A}\right)\right\}^{1/2}, \tag{4.30}$$

where A_0 is an abbreviation for $A(\frac{1}{2})$.

We introduce

$$\beta = A_0 - \frac{\alpha^2}{3}\frac{1}{A_0} \tag{4.31}$$

and

$$A = \frac{\beta}{2}+\left(\frac{\alpha^2}{3}+\frac{\beta^2}{4}\right)^{1/2}\sin\varphi, \tag{4.32}$$

where

$$\varphi_1 \leqslant \varphi \leqslant \pi/2. \tag{4.33}$$

In (4.33) $|\varphi_1| < \pi/2$. It appears that for some values of A_0, φ may become negative.

Integrating (4.30) we find

$$\frac{\beta}{2}\left(\frac{\pi}{2}-\varphi_1\right)+\mu\cos\varphi_1+\frac{\alpha}{\sqrt{3}}\log\left\{\frac{\frac{\beta}{2}+\mu-\frac{\alpha}{\sqrt{3}}}{\frac{\beta}{2}+\mu+\frac{\alpha}{\sqrt{3}}}\cdot\frac{\frac{\beta}{2}\tan\frac{\varphi_1}{2}+\mu+\frac{\alpha}{\sqrt{3}}}{\frac{\beta}{2}\tan\frac{\varphi_1}{2}+\mu-\frac{\alpha}{\sqrt{3}}}\right\}=\frac{2}{\sqrt{3}}\sqrt{P}(\tfrac{1}{2}-d), \tag{4.34}$$

where φ_1 is defined by

$$\varphi_1 = \arcsin\frac{A_1-\beta/2}{\mu}, \quad \left(>-\frac{\pi}{2}\right) \tag{4.35}$$

and we have simplified the formulae by introducing the abbreviation

$$\mu = \left(\frac{\alpha^2}{3}+\frac{\beta^2}{4}\right)^{1/2}. \tag{4.36}$$

The value of φ_1 becomes negative for $A_1 < \beta/2$ or

$$A_0 > A_1 + \left(A_1^2 + \frac{\alpha^2}{3}\right)^{1/2}. \tag{4.37}$$

Equation (4.34) is a relation between the unknowns β, P and d. To solve the complete problem two other relations have to be derived. In the interval $0 \leq z \leq d$, the solution of (4.7) and (4.2) is of the form

$$u = B \sin\left(\frac{P}{A_1^2-\alpha^2}\right)^{1/2} z, \tag{4.38}$$

where B is unknown. We may find B from the displacement at $z = d$. We have

$$u(d) = B \sin\left(\frac{P}{A_1^2-\alpha^2}\right)^{1/2} d = (2P)^{-1/2} A_1^{-1/2}(A_1^2-\alpha^2), \tag{4.39}$$

where (4.27) has been used, while the area A has to be continuous. We eliminate B from (4.38) and (4.39) and obtain

$$u(z) = (2P)^{-1/2}\frac{A_1^2-\alpha^2}{A_1^{1/2}}\frac{\sin\left(\frac{P}{A_1^2-\alpha^2}\right)^{1/2} z}{\sin\left(\frac{P}{A_1^2-\alpha^2}\right)^{1/2} d}, 0 \leq z \leq d. \tag{4.40}$$

It might be surprising that we can find an explicit expression for the amplitude of the homogeneous problem (4.7), (4.2), but (4.39) is nothing else then a normalisation for u, based upon the equality (4.20). The derivative of $u(z)$ with respect to z at $z = d$ is found from (4.40) to be

$$u'(d) = \frac{1}{\sqrt{2}}\left(\frac{A_1^2-\alpha^2}{A_1}\right)^{1/2} \cot\left(\frac{P}{A_1^2-\alpha^2}\right)^{1/2} d. \tag{4.41}$$

A calculation of the same derivative from (4.27) yields

$$u'(d) = \sqrt{\left(\frac{3}{2}\right)}\, A_1^{-1/2}\left\{\beta A_1 - A_1^2 + \frac{\alpha^2}{3}\right\}^{1/2}. \tag{4.42}$$

In fact, the expression (4.41) is $u'(d_-)$, while (4.42) gives $u'(d_+)$. The continuity of the derivative of u at $z = d$ yields the second relation between β, P and d. It is

$$\sqrt{3}\frac{\mu}{(A_1^2-\alpha^2)^{1/2}}\cos\varphi_1 = \cot\left(\frac{P}{A_1^2-\alpha^2}\right)^{1/2} d. \tag{4.43}$$

The third relation is found from

$$A_1 d + \int_d^{1/2} A \, \mathrm{d}z = V/2. \tag{4.44}$$

We find for this equation

$$\tfrac{3}{2}\mu^2\left(\frac{\pi}{2}-\varphi_1\right) + \beta\mu\cos\varphi_1 + \tfrac{1}{2}\mu^2\cos\varphi_1\sin\varphi_1 = \frac{2}{\sqrt{3}}\sqrt{P}\left(\frac{V}{2}-A_1 d\right). \tag{4.45}$$

The eqns (4.34), (4.43) and (4.45) are sufficient for the determination of β, P and d. Once these quantities determined, we can calculate the shape of the column by integrating (4.30). We find

$$A = A_1, 0 \leqslant z \leqslant d,$$

$$\frac{\beta}{2}(\varphi - \varphi_1) - \mu(\cos\varphi - \cos\varphi_1) + \frac{\alpha}{\sqrt{3}}\log\left\{\frac{\frac{\beta}{2}\tan\frac{\varphi}{2} + \mu - \frac{\alpha}{\sqrt{3}}}{\frac{\beta}{2}\tan\frac{\varphi}{2} + \mu + \frac{\alpha}{\sqrt{3}}} \cdot \frac{\frac{\beta}{2}\tan\frac{\varphi_1}{3} + \mu + \frac{\alpha}{\sqrt{3}}}{\frac{\beta}{2}\tan\frac{\varphi_1}{2} + \mu - \frac{\alpha}{\sqrt{3}}}\right\}$$

$$= \frac{2}{\sqrt{3}}\sqrt{P}(z - d), \quad d \leqslant z \leqslant \tfrac{1}{2}, \qquad (4.46)$$

$$A = \frac{\beta}{2} + \mu \sin\varphi, \quad d \leqslant z \leqslant \tfrac{1}{2}.$$

(b) *The general case*

We proceed by considering the case, in which both of the areas may vary. The functional becomes here

$$\int_0^1 \left[A - cA_1 + \lambda_1\left(\mu'' + \frac{P}{A^2 - sA_1^2}u\right) + \lambda_2(q_1^2 - A + A_1) + \lambda_3(q_2^2 - A_1 + p_1)\right] dz. \qquad (4.47)$$

In (4.47) c is an unknown constant, to be determined and A and A_1 are subject to the inequalities

$$A \geqslant A_1, \quad A_1 \geqslant p_1, \qquad (4.48)$$

while we assume $s < 1$.

Taking small variations of the field quantities A, A_1, u, q_1 and q_2, integrating one term by parts and using the boundary conditions (4.2), we obtain

$$\lambda_1'' + \frac{P}{A^2 - sA_1^2}\lambda_1 = 0, \lambda_1(0) = \lambda_1(1) = 0, \qquad (4.50)$$

$$\lambda_2 q_1 = 0, \qquad (4.51)$$

$$\lambda_3 q_2 = 0, \qquad (4.52)$$

$$\lambda_1\frac{2PuA}{(A^2 - sA_1^2)^2} + \lambda_2 - 1 = 0, \qquad (4.53)$$

$$\lambda_1\frac{2PusA_2}{(A^2 - sA_1^2)^2} + \lambda_2 - \lambda_3 - c = 0. \qquad (4.54)$$

From (4.50) it follows that again we may put

$$\lambda_1 = u, \qquad (4.55)$$

and we find from (4.53) and (4.54) the equivalent equations

$$\frac{2Pu^2A}{(A^2 - sA_1^2)^2} = 1 - \lambda_2 \qquad (4.56)$$

$$cA - sA_1 + (\lambda_3 - \lambda_2)A + s\lambda_2 A_1 = 0. \qquad (4.57)$$

We shall show by a discussion, similar to the discussion given in Section 4a, that we have

$$A = A_1 = p_1, \quad 0 \leq z \leq d_1, \tag{4.58}$$

$$A > A_1 = p_1, \quad d_1 \leq z \leq d_2, \tag{4.59}$$

$$A > A_1 > p_1, \quad d_2 \leq z \leq \tfrac{1}{2}, \tag{4.60}$$

where d_1 and d_2 are unknowns. In the interval $d_2 \leq z \leq \frac{1}{2}$ the equality

$$A = \frac{s}{c} A_1 \tag{4.61}$$

holds. From (4.48) it follows that

$$s > c. \tag{4.62}$$

The value of the functions λ_2 and λ_3 at $z = 0$ are

$$\lambda_2(0) = 1; \quad \lambda_3(0) = 1 - c. \tag{4.63}$$

If we substract eqn (4.54) from (4.53), and apply the result to the interval $d_2 \leq z \leq \frac{1}{2}$, where $\lambda_2 = \lambda_3 = 0$, we find

$$1 - c = \frac{2Pu^2}{A^2 - sA_1^2} (>0), \tag{4.64}$$

thus both λ_2 and λ_3 start with positive values. At $z = d_1$, one of these functions become zero. Because at this point the equation

$$c - s + (\lambda_3 - \lambda_2) + s\lambda_2 = 0 \tag{4.65}$$

holds, we see that

$$\lambda_2(d_1) = 0,$$

on account of (4.62). (If we should assume $\lambda_3(d_1) = 0$, we find from (4.65) that $\lambda_2(d_1) < 0$ and then it has passed a zero-point). From (4.66), (4.59) follows.

We conclude that in the interval $0 \leq z \leq d_1$, the bar is purely cylindrical, in the interval $d_1 \leq z \leq d_2$ the problem is equivalent to that of Section 4a and in the middle of the bar, the equation for the optimal shape is similar to that of the homogenous bar.

A complete discussion of this problem goes along the same lines as the discussion, given in Section 4a. However, it is more complex, because there are more unknowns, and the optimal shape has more arcs. Therefore, we shall not treat it in detail, but we shall point that we may expect it to have a unique solution.

The unknowns in this problem are: P, d_1, d_2, c, B (the coefficient of u on $0 \leq z \leq d_1$, satisfying the boundary condition at $z = 0$), C_1, C_2 (the coefficients of u on $d_1 \leq z \leq d_2$), and D (the coefficient of u on $d_2 \leq z \leq \frac{1}{2}$, satisfying the condition at $z = \frac{1}{2}$). Altogether there are eight unknowns. We have the following equations: at $z = d_1$: three, resulting from the continuity of u and u' and the value of u from $\lambda_3 = 0$; at $z = d_2$: three, the corresponding equations; further the values of V_1 and V_2. Thus the same number of equations as there are unknowns. Note that the constant c cannot be determined *à priori*. It is an unknown in the problem.

5. PROOF THAT THE STATIONARY VALUE GIVES A MAXIMUM FOR P

We only discuss the problem of Section 4a. We have to change some of the notations. Let us denote the optimal P by P^*, the corresponding displacement and area by u^* and A^*, respectively. Then we have

$$u^{*\prime\prime} + \frac{P^*}{A^{*2} - \alpha^2} u^* = 0, \; u^*(0) = u^*(1) = 0. \tag{5.1}$$

We know that

$$P^* = \frac{\int_0^1 (u^{*\prime})^2\,dz}{\int_0^1 \frac{u^{*2}}{A^{*2}-\alpha^2}\,dz} = \min_u \frac{\int_0^1 (u')^2\,dz}{\int_0^1 \frac{u^2}{A^{*2}-\alpha^2}\,dz}, \tag{5.2}$$

where the minimum is taken over the class of all admissible functions u.

Now consider the buckling problem of a bar with area $\bar{A}$. The corresponding buckling load is $\bar{P}$ and the displacement $\bar{u}$. We have

$$\bar{u}'' + \frac{\bar{P}}{\bar{A}^2-\alpha^2}\bar{u} = 0, \bar{u}(0) = \bar{u}(1) = 0,$$

and again the following property holds

$$\bar{P} = \frac{\int_0^1 (\bar{u}')^2\,dz}{\int_0^1 \frac{\bar{u}^2}{\bar{A}^2-\alpha^2}\,dz} = \min_u \frac{\int_0^1 (u')^2\,dz}{\int_0^1 \frac{u^2}{\bar{A}^2-\alpha^2}\,dz}. \tag{5.3}$$

We proceed by putting u^* into the expression (5.3). Then we obtain

$$\bar{P} \leq \frac{\int_0^1 (u^{*\prime})^2\,dz}{\int_0^1 \frac{u^{*2}}{\bar{A}^2-\alpha^2}\,dz}. \tag{5.4}$$

Now we have to compare the first quotient in (5.2) with (5.4). It is obvious that

$$P^* > \bar{P}, \tag{5.5}$$

if

$$J \equiv \int_0^1 (u^*)^2\,dz \left\{\frac{1}{A^{*2}-\alpha^2} - \frac{1}{(A^*+B)^2-\alpha^2}\right\} < 0, \tag{5.6}$$

where we have put

$$\bar{A} = A^* + B. \tag{5.7}$$

Because for both $\bar{A}$ and A^* the same constraint (4.3) holds, we have

$$\int_0^1 B\,dz = 0. \tag{5.8}$$

The inequality (5.5) is strict, because (5.6) is strict. To prove (5.6), we split up the integral into three integrals over the intervals $(0, d)$, $(d, 1-d)$ and $(1-d, 1)$. We have (see (4.27) and (4.40))

$$u^{*2} = t\frac{(A^{*2}-\alpha^2)^2}{A^*}, \quad d \leq z \leq 1-d, \tag{5.9}$$

$$u^{*2} \leq t\frac{(A_1{}^2-\alpha^2)^2}{A_1}, \quad 0 \leq z \leq d; \quad 1-d \leq z \leq 1, \tag{5.10}$$

$$\text{with } t > 0. \tag{5.11}$$

With (5.9) and (5.10) the integral (5.6) becomes

$$J \leq 2t\int_0^1 B\,\mathrm{d}z - t\left[\int_0^d B^2\frac{3A_1^2+2A_1B+\alpha^2}{A_1[(A_1+B)^2-\alpha^2]}\mathrm{d}z\right.$$
$$\left.+\int_d^{1-d} B^2\frac{3A^{*2}+2A^*B+\alpha^2}{A^*[(A^*+B)^2-\alpha^2]}\mathrm{d}z+\int_{1-d}^1 B^2\frac{3A_1^2+2A_1B+\alpha^2}{A_1[(A_1+B)^2-\alpha^2]}\right]\mathrm{d}z.$$

Because A_1 is the minimum value of A, we have in $(0, d)$ and in $(1-d, 1)$

$$B>0. \tag{5.13}$$

In $(d, 1-d)$, B may be negative, but there exists the constraint

$$(A^*+B)^2>\alpha^2. \tag{5.14}$$

From (5.14) we have

$$3A^{*2}+2A^*B+\alpha^2=(A^*+B)^2+2A^{*2}-B^2-\alpha^2+2\alpha^2>2A^{*2}-B^2+2\alpha^2$$
$$>2A^{*2}+2\alpha^2-(A^{*2}+\alpha^2-2A^*\alpha)=A^{*2}+2A^*\alpha+\alpha^2=(A^*+\alpha)^2>0.$$

From (5.11), (5.13) and (5.15), the inequality (5.6) follows and we may conclude that (5.5) holds.

6. GENERALIZATIONS

We have worked out the problem for the buckling of the hinged column in some detail. In this section we shall indicate, how we can generalize the method to other boundary conditions. We restrict our discussions to the compound bar with an inner component of fixed cross section. The differential equation for the displacement is

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\{(A^2-\alpha^2)u''\}+Pu''=0, \tag{6.1}$$

written in non-dimensional form (see (3.7), (2.10) and (4.6)). The variational principle is here

$$\delta\int_0^1 [A+\lambda_1\{((A^2-\alpha^2)u'')''+Pu''\}+\lambda_2(q^2-A+A_1)]\,\mathrm{d}z=0. \tag{6.2}$$

If the boundary value problem is self-adjoint (and we shall restrict ourselves to these problems) we find after integration by parts for λ_1:

$$((A^2-\alpha^2)\lambda_1'')''+P\lambda_1''=0, \tag{6.3}$$

with the same boundary conditions for λ_1 as for u. Hence we may put

$$\lambda_1=cu, \tag{6.4}$$

where c is a constant that will be chosen appropriately. We also find

$$\lambda_2 q=0, \tag{6.5}$$

and

$$-2\lambda_1''Au''+1-\lambda_2=0. \tag{6.6}$$

We may integrate (6.1) twice and obtain

$$(A^2-\alpha^2)u''+Pu=a_1P+a_2Pz, \tag{6.7}$$

where a_1 and a_2 are constants to be determined from the boundary conditions. Now we introduce the function u_0 by

$$u = u_0 + a_1 + a_2 z \tag{6.8}$$

that satisfies

$$u_0'' + \frac{P}{A^2 - \alpha^2} u_0 = 0; \quad u_0'' = u''. \tag{6.9}$$

With the choice

$$c = -\frac{1}{P} \tag{6.10}$$

we may write (6.6), on account of (6.4), and (6.9), in the form

$$\frac{2PAu_0^2}{(A^2-\alpha^2)^2} + \lambda_2 - 1 = 0, \tag{6.11}$$

a form, equivalent, but not equal to (4.21).

Now the different boundary condition problems may be treated. For the clamped-clamped case we find that the optimal shape consists of five arcs, symmetrically situated with respect to $z = \frac{1}{2}$.

It appears that at $z = 0$, the area of the cross section has a relative maximum. In this problem there are nine unknowns: a_1, ($a_2 = 0$), d_1, d_2, P, the amplitudes B_1, B_2 of the displacement in the interval (d_1, d_2), an integration constant b_1 for the equation in the interval $(0, d_1)$, the corresponding integration constant b_2 in the interval $(d_2, \frac{1}{2})$ and the value $u_0(\frac{1}{2})$. There are also nine equations: at $z = 0$: one for the boundary condition, at $z = d_1$: three, resulting from the continuity of u and u' and the value of u, at $z = d_2$: three corresponding equations, at $z = \frac{1}{2}$: the condition $u' = 0$, and the specification of the volume. We shall not go into further details.

7. CONCLUSIONS AND RESULTS

We have reduced the analytical problem to the solution of a few transcendental equations, such as (4.34), (4.43) and (4.45). From these we find the basic parameters and then we may calculate all other quanties as the shape of the column (see 4.46) and the stress distribution.

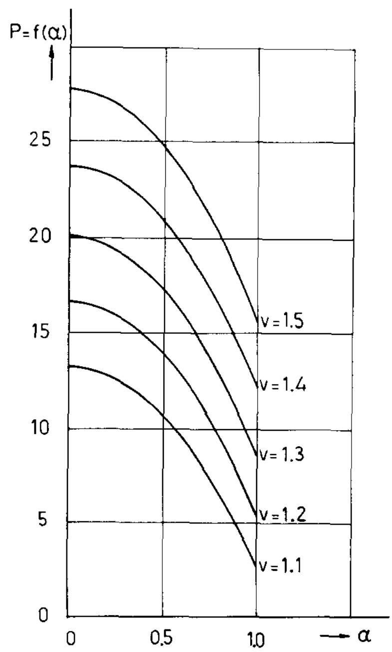

Fig. 2. Non-dimensional buckling load as a function of α for different values of V.

Fig. 3. The cross section of a column of optimal strength.

In Fig. 2 we have plotted the non-dimensional P as a function of α for different values of V. The figure has reference to the case, treated in Section 4a, with $k_1 = k_2$ and the minimum of A equal to A_1. For our numerical calculations we have taken $A_1 = 1$. This means that we have grouped the non-dimensional quantities into dimensionless parameters. For arbitrary A_1 we have to transform our quantities as follows

$$P \to PA_1^2,\ V \to VA_1,\ \alpha \to \alpha A_1,\ \beta \to \beta A_1. \tag{7.1}$$

For the interpretation of Fig. 2 we write the real buckling load P as (see (4.9) and (7.1))

$$P = \bar{P}E_2k_2l^2\bar{A}_1^2 = \bar{P}\frac{E_2k_2}{l^2}A_1^2 = \bar{P}\frac{E_2J_1}{l^2} = \bar{P}\frac{E_1J_1}{l^2}\left(\frac{E_2}{E_1}\right) = \bar{P}\frac{E_1J_1}{l^2}\frac{1}{1-s}. \tag{7.2}$$

If $s(=\alpha^2/A^2) = 0$ we have the homogeneous bar. In this case we may take for the volume the value A_1l and then find

$$\bar{P} = \pi^2, \left(\frac{V}{A_1l} = 1, s = 0\right). \tag{7.3}$$

As can be expected, $\bar{P}$ decreases with increasing s, because this corresponds with decreasing bending stiffness. For $s = 1$ we find the buckling load of the (optimum) hollow beam ($E_1 = 0$). As long as $V/A_1l > 1$, this buckling load remains positive, because, for the case under discussion ($A \geqslant A_1$), the value of d tends to zero like

$$d \approx (1-s)\{(3P)^{1/2}\mu \cos \varphi_1\}, \tag{7.4}$$

as can easily be found from (4.41) and (4.42). For the hollow beam (4.34) and (4.45) provide two equations for the determination of P and β, if d is taken equal to zero in these equations. In Fig. 3 we show the cross section A as a function of z for the values $\alpha = 0.5$; $V = 1, 3$.

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